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A NOTE ON A STOCHASTIC PRODUCTION-MAXIMIZING TRANSPORTATION PROBLEM

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ABSTRACT

A stochastic production-maximizing problem with transportation constraints is considered where the production rates, R_{ij}, of man i − job j combinations are random variables rather than constants. It is shown that for the family of Weibull distributions (of which the Exponential is a special case) with scale parameters λ_{ij} and shape parameter β, the plan that maximizes the expected rate of the entire line is obtained by solving a deterministic fixed charge transportation problem with no linear costs and with "set-up" cost matrix $||\lambda_{ij}||$.

The Time-Minimizing Transportation Problem (TMTP) was treated by Barsov (1959) [2] and again by Hammer (1969) [4] and may be stated as follows: Given a set of m origins and n destinations, where there are a_i $(i=1,2,\ldots,m)$ units available at the ith origin and b_j $(j=1,2,\ldots,n)$ units required at the jth destination (such that, $\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j$), find a set of nonnegative variables x_{ij} $(i=1,2,\ldots,m;$ $j=1,2,\ldots,n)$ satisfying the (classical transportation-type) constraints

$$\sum_{j=1}^{n} x_{ij}=a_i \qquad (i=1,2,\ldots,m)$$

(1)

$$\sum_{i=1}^{m} x_{ij}=b_j \qquad (j=1,2,\ldots,n)$$

and minimizing the *greatest* of the given nonnegative numbers t_{ij} for which $x_{ij}>0$.

The t_{ij} may be interpreted as the time required to transport a positive load x_{ij} (however big or small) from the ith origin to the jth destination. Thus the problem is to find a transportation plan which makes the most time-consuming trip as short as possible.

In a production context the analogous problem will be to consider the *rate* of production, R_{ij}, instead of the time t_{ij}, and to seek a production plan $X=\{x_{ij}\}$ satisfying (1) and maximizing the *smallest* of the given nonnegative rates R_{ij} for which $x_{ij}>0$. The R_{ij}'s are now interpreted as the rate of production (on a production line, say) of a man belonging to group (origin) i when he is assigned to job (destination) j. As above, there are a_i men available in the ith group and b_j men required for the jth job.

For any production plan X that satisfies (1) let $A_X = \{(ij) | x_{ij} > 0\}$. For any such plan, the rate of production of the entire line, R, will be given by

$$R = \underset{(ij)\epsilon A_X}{\text{Minimum}} (R_{ij}) \tag{2}$$

and the problem is then to find a plan for which R is as large as possible.

Now, suppose that for each man i — job j combination, the corresponding R_{ij} is not a constant, but a continuous nonnegative random variable with distribution function $F_{ij}(\cdot)$. This implies that, for any plan X, R (as given by (2)) is also a random variable. Our objective then is to find a production plan that will maximize the *expected* rate of production of the line, i.e., we seek a plan X satisfying (1) so as to achieve

$$\underset{X}{\text{Max}}\{E[R]\} = \underset{X}{\text{Max}}\{E[\underset{(ij)\epsilon A_X}{\text{Minimum}} (R_{ij})]\}. \tag{3}$$

Assuming that the R_{ij}'s are independent random variables with finite means, the distribution function of R, $F_R(\cdot)$, is found to be

$$F_R(r) = 1 - \prod_{(ij)\epsilon A_X} [1 - F_{ij}(r)],$$

and the expected rate of production is given by

$$E[R] = \int_0^\infty \left\{ \prod_{(ij)\epsilon A_X} [1 - F_{ij}(r)] \right\} dr.$$

Now consider the family of Weibull distributions where R_{ij} has a scale parameter $\lambda_{ij} > 0$ and shape parameter $\beta > 0$ (equal for all man-job combinations). In this case, the distribution function of R_{ij} is

$$F_{ij}(r) = 1 - \exp(-\lambda_{ij} r^\beta), \qquad r \geqslant 0. \tag{4}$$

We consider also the following "Fixed Charge Transportation Problem" (FCTP) with no linear costs: Given $\lambda_{ij} > 0$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ find a plan X satisfying (1) so as to achieve

$$\text{Min} \left\{ \sum_{(ij)\epsilon A_X} \lambda_{ij} \right\}. \tag{5}$$

We now show the following:

THEOREM: The solution to the stochastic production-maximizing transportation problem (3) is given by the solution of the FCTP (5).

PROOF: For any given plan X we obtain:

$$E[R]=\int_0^\infty \exp\Big\{-\Big(\sum_{(ij)\epsilon A_X}\lambda_{ij}\Big)r^\beta\Big\}dr = \left(\frac{1}{\sum_{(ij)\epsilon A_X}\lambda_{ij}}\right)^{1/\beta}\Gamma\Big(\frac{1}{\beta}+1\Big), \tag{6}$$

where $\Gamma(\cdot)$ denotes the Gamma function. It is clear that $E[R]$ in (6) is maximized whenever $\sum_{(ij)\epsilon A_X}\lambda_{ij}$ is minimized. This completes the proof.

By letting $\beta=1$ in (4) it is readily seen that the exponential family of distributions is a special case of the family of Weibull distributions. Note also that if we let $m=n$ and $a_i=b_j=1$ for all i and j then the deterministic and stochastic production-maximizing problems are transformed, respectively, into the classical [3] and stochastic [6] bottleneck assignment problems, whereas the FCTP [1] is transformed into the assignment problem.

In general, fixed charge problems have proven difficult to solve, primarily because each extreme point (here a basic solution of (1)) of the convex set of feasible solutions is a local optima. In our case, however, a direct way to solve the FCTP would be to enlarge it into an assignment problem of order

$$\Big(\sum_{i=1}^m a_i\Big)\times\Big(\sum_{j=1}^n b_j\Big).$$

Another approach could be to formulate the FCTP as an all-integer linear program [1]. A third method would employ a branch and bound algorithm as presented in [5]; however, for large problems all of the above methods would eventually become inefficient, and an approximative procedure (such as the one suggested in [1]) seems to be more practical. Additional references for approximative methods may be found in [5].

In summary, we have shown that the plan that maximizes the expected production rate of the entire line in a randomized production-maximizing transportation problem can be found by solving a deterministic fixed charge transportation problem with no linear costs and with fixed-charge cost matrix $\|\lambda_{ij}\|$ whose entries are the scale parameters of the random variables R_{ij}.

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